Where does it go? Displacement and velocity as vectors

1. A boat is **sailing** on open sea...

2. The **speed** is 10.0 knots (a knot is 1 nautical mile per hour, 1.852 km/h) and is heading 045 degrees (this means it is pointing nordest).

3. Where is the boat after 10 hours sailing, if nothing changes?

4. If the boat was heading in another direction, the displacement would be the same (100 miles)... but the final point would be on any of the points on a circumference centered on the starting point...

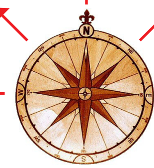
5. Velocity *can't be treated as having only* **magnitude** (a numerical value...). It also has **direction** (where does it point to?).

6. The same is true for **displacement**: it has **magnitude** (how many miles from the starting point?) and **direction** (in what line from the starting point is the final point?).

magnitude of displacement = 10 miles/h × 10 h = 100 miles

$$\Delta\vec{r} = \vec{v} \times \Delta t$$

displacement = velocity × time interval



000 degrees (or 360) to north

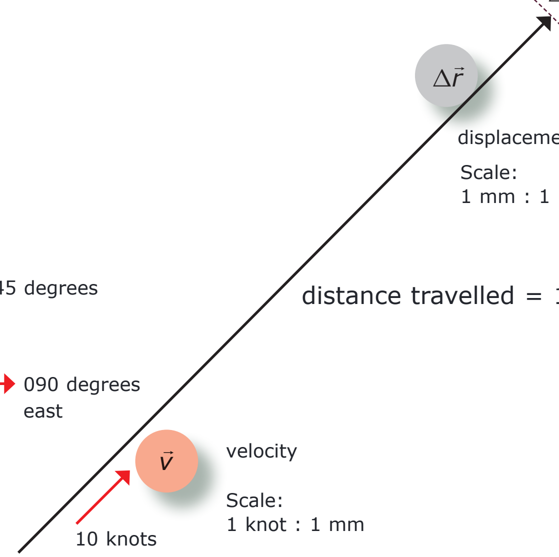
315 degrees

045 degrees

270 degrees west

090 degrees east

180 degrees south



$\Delta\vec{r}$

displacement

Scale: 1 mm : 1 mile

distance travelled = 100 miles

$\vec{v}$

velocity

10 knots

Scale: 1 knot : 1 mm

(20 mm)

How big is this velocity, if the scale is the same of the above velocity?

Where would the boat be after 5 hours sailing in this direction?

☐ OK ☐ I need to do this again... ☐ I need tutor support

Where does it go? Changing velocity...

1. A boat is sailing on open sea...

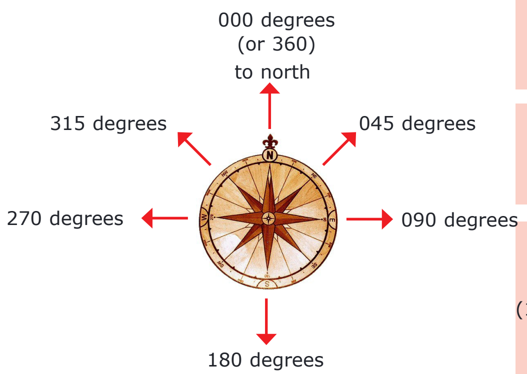
2. Velocity: heading 045 degrees, speed 5.0 knots. Sailing this velocity for 1 hour, then...

3. ... new velocity: heading 090 degrees, speed 5.0 knots. Sailing this velocity for 2 hours, then...

4. ... new velocity: heading 000 degrees, speed 7.0 knots. Sailing this velocity for 2 hours, then...

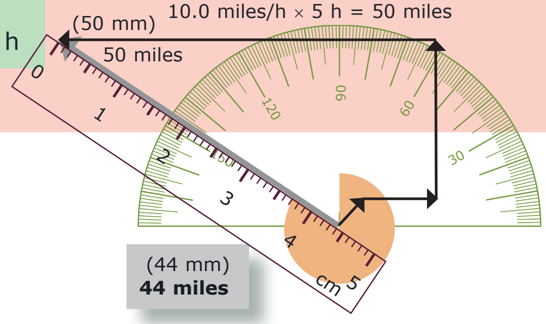
5. ... new velocity: heading 270 degrees, speed 10.0 knots. Sailing this velocity for 5 hours...

6. How far from the starting point is the final point? Where is it? What was the total distance travelled by the boat?



displacement = velocity × time interval

velocity	time of travel	displacement	distance travelled
Scale: 2 mm : 1 knot		Scale: 1 mm : 1 mile	
(10 mm) 5.0 knots	1 h	(5 mm) 5 miles 5.0 miles/h × 1 h = 5 miles	5 miles (5 miles until now...)
5.0 knots (10 mm)	2 h	5.0 miles/h × 2 h = 10 miles (10 mm) 10 miles	10 miles (15 miles until now...)
7.0 knots (14 mm)	1.5 h	(21 mm) 21 miles 7.0 miles/h × 1.5 h = 21 miles	21 miles (36 miles until now...)
10.0 knots (20 mm)	5 h	10.0 miles/h × 5 h = 50 miles (50 mm) 50 miles	50 miles (86 miles until now...)



(44 mm)
44 miles

86 miles (total)

angle with north = 146º – 90º = 56º

angle with north, measured from north = 360º – 56º = 304º

A model of the motion of the boat

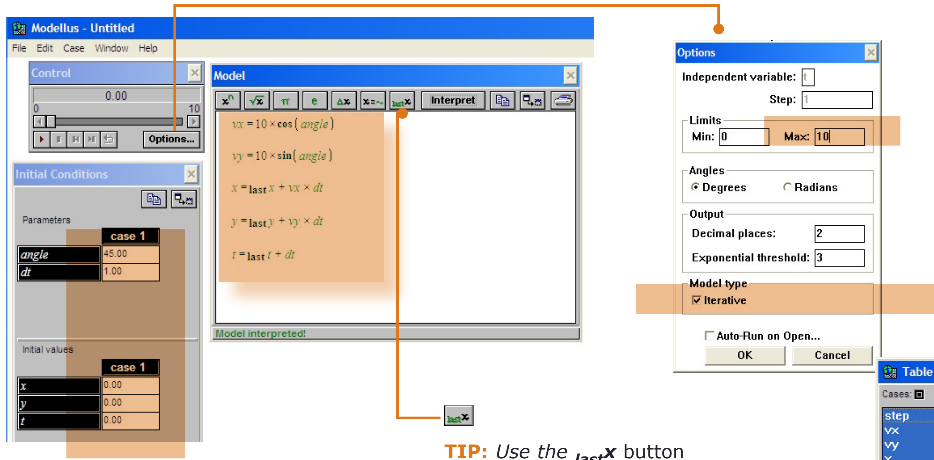
1. Create the following model in Modellus (don't forget to select **Iterative Model** and **Max** number of steps = 10 using the Options... button on the Control

2. Create a **particle** (the **boat**...) with coordinates *x* and *y* and a vector (the displacement of the boat...) with components *x* and *y*.

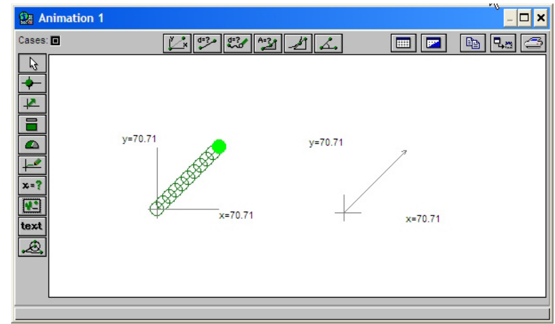
3. Run the model and check how it works...

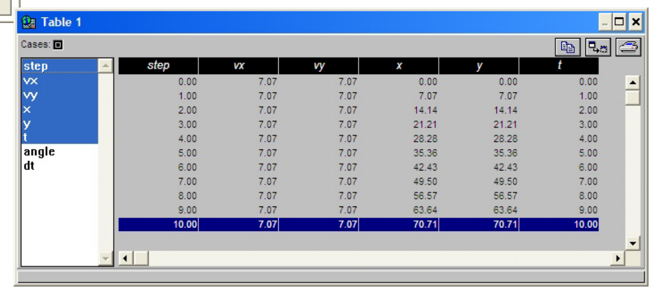
4. Create a table and inspect all the values of the different variables.

5. How far did the boat moved? What are the position of the boat if the reference frame has its origin on the departure point?



TIP: *Use the* $_{last}x$ button to get the symbol *last before each variable*





step	vx	vy	x	y	t
0.00	7.07	7.07	0.00	0.00	0.00
1.00	7.07	7.07	7.07	7.07	1.00
2.00	7.07	7.07	14.14	14.14	2.00
3.00	7.07	7.07	21.21	21.21	3.00
4.00	7.07	7.07	28.28	28.28	4.00
5.00	7.07	7.07	35.36	35.36	5.00
6.00	7.07	7.07	42.43	42.43	6.00
7.00	7.07	7.07	49.50	49.50	7.00
8.00	7.07	7.07	56.57	56.57	8.00
9.00	7.07	7.07	63.64	63.64	9.00
10.00	7.07	7.07	70.71	70.71	10.00

$vx = 10 \times \cos(angle)$ horizontal component of the velocity...

$vy = 10 \times \sin(angle)$ vertical component of the velocity...

$x = \mathrm{last}\,x + vx \times dt$ new value of the *x* coordinate = last value of *x* multiplied by a time interval

$y = \mathrm{last}\,y + vy \times dt$ new value of the *y* coordinate = last value of *y* multiplied by a time interval

$t = \mathrm{last}\,t + dt$ new value of time = last value of *t* + time interval

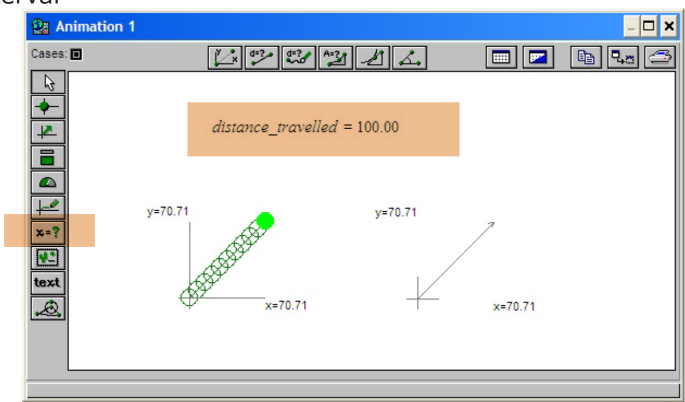
After using the above model, add this new iteration:

$$distance\_travelled = \mathrm{last}\,distance\_travelled + \sqrt{\left(\Delta x^2 + \Delta y^2\right)}$$

Check its value on the table or place a digital meter on the Animation Window to see how it changes (the square root and the delta/change symbols can be obtained clicking on the buttons $\sqrt{x}$ and Δx on the Model Window).

The variable *distance_travelled* (be sure not to make spelling mistakes!) "accumulates" the change in the magnitude of displacement in each iteration.

TIP: *Use* the digital meter to see values of variables on the Animation Window.



Moving the boat with constant speed and different velocity...

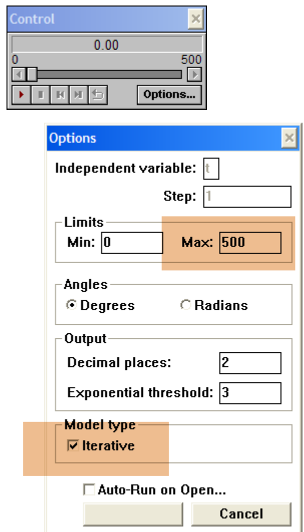
1. Let's now create a model with a control bar that will be used to change the angle of the velocity...

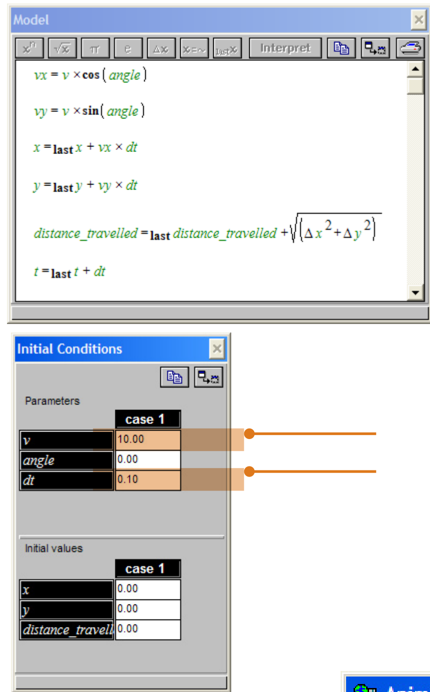
2. Re-use the previous model or create the following model on Modellus (don't forget to select **Iterative Model** and **Max** number of steps = 500 using the Options... button on the Control

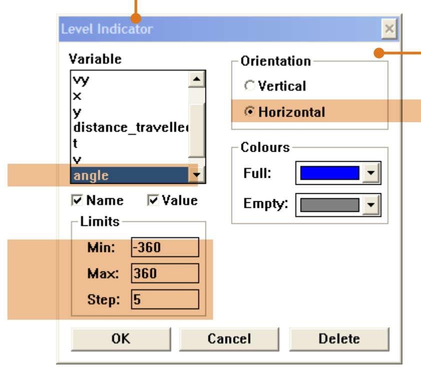
3. Besides the particle (the boat...) with coordinates *x* and *y*, place a "level indicator"/"horizontal bar" on the Animation Window. Give adequate properties to this object.

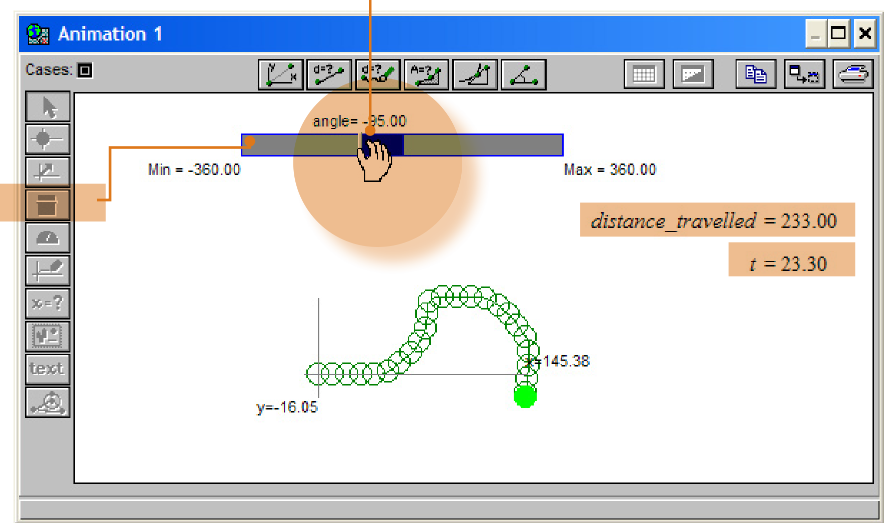
4. Run the model and change the value of the angle.

5. You can also place digital meters to check the distance travelled by the boat and the time spent.









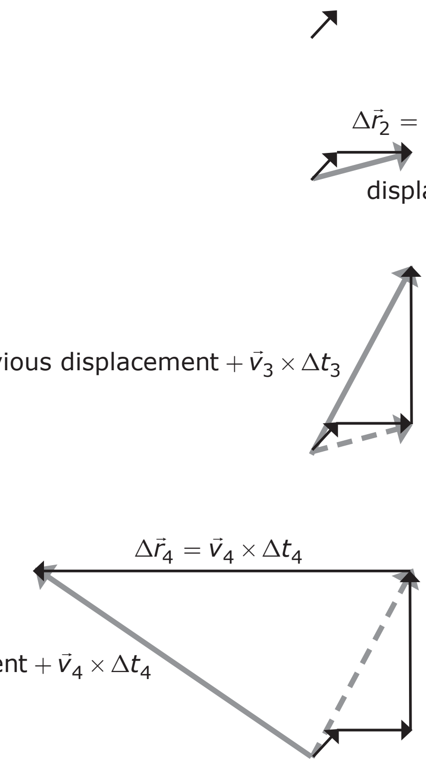
- Move the "level indicator" to change the angle that velocity makes with the horizontal...
- Try to move the particle in order to return to the initial point...
- See how distance travelled changes...
- Is velocity changing? Why?
- With constant speed, it is easy to compute distance travelled: we just need to multiply the speed (the magnitude of velocity) by the time interval. Check this relation...

Iterative solution of the equations of motion

1. When an object is moving with **constant velocity**, after a time interval of Δt we know that its **displacement** will be $\vec{v} \times \Delta t$ ...

2. Displacement can be computed from previous displacement plus displacement on current time interval...

3. This is the **iterative solution of the equation of motion**: *each new value of the positional vector is computed from the previous value plus change during the current interval.*



If Δt is allways the same, we can write...

For the displacement vector:

$$\vec{r}_{t+\Delta t} = \vec{r}_t + \vec{v}_t \times \Delta t$$

For each component:

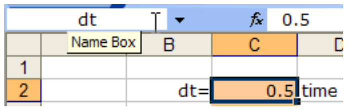
$x = \text{last}\, x + vx \times dt$

$y = \text{last}\, y + vy \times dt$

$\vec{r}$ represents the displacement from the origin O (i.e., the positional vector on the Oxy reference frame)

Numerical/iterative solution of the equations of motion in a spreadsheet

TIPS: *To define a name on a cell, such as dt for cell C2, place the cursor on the cell and write dt (the name) on the name box. Define also cells C3 and C4 as x0 and y0, the initial values of the coordinates.*



=C7*COS(D7*PI()/180)

=C7*SIN(D7*PI()/180)

TIPS: *Excel represents the number π as PI(). Angles are expressed in radians. To convert degrees to radians, the conversion factor is π/180)*

=x0 =y0 *These are the inital values of the coordinates*

=G7+E7*dt

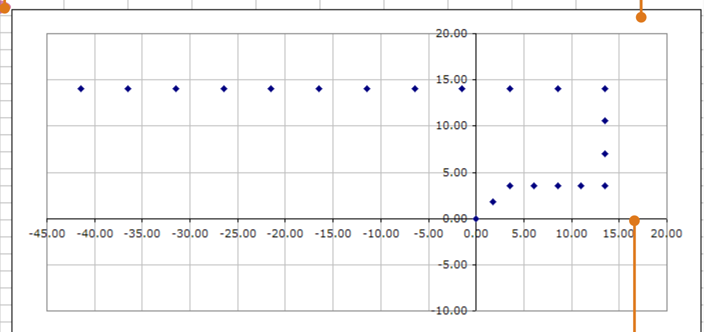
=H7+F7*dt

The next value of each coordinate is the previous value plus change (velocity component multiplied by time interval)

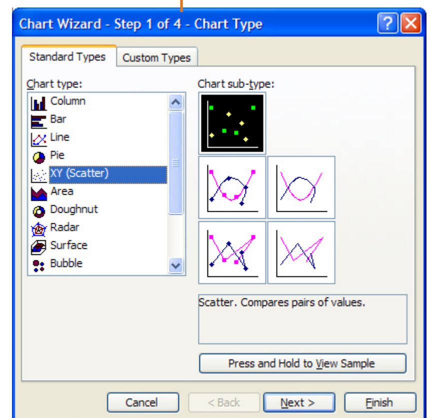
	A	B	C	D	E	F	G	H
1								
2		dt=	0.5	time step in seconds				
3		x0=	0	initial coordinates				
4		y0=	0					
5								
6		t	speed	angle	vx	vy	x	y
7		0.00	5.00	45.00	3.54	3.54	0.00	0.00
8		0.50	5.00	45.00	3.54	3.54	1.77	1.77
9		1.00	5.00	0.00	5.00	0.00	3.54	3.54
10		1.50	5.00	0.00	5.00	0.00	6.04	3.54
11		2.00	5.00	0.00	5.00	0.00	8.54	3.54
12		2.50	5.00	0.00	5.00	0.00	11.04	3.54
13		3.00	7.00	90.00	0.00	7.00	13.54	3.54
14		3.50	7.00	90.00	0.00	7.00	13.54	7.04
15		4.00	7.00	90.00	0.00	7.00	13.54	10.54
16		4.50	10.00	180.00	-10.00	0.00	13.54	14.04
17		5.00	10.00	180.00	-10.00	0.00	8.54	14.04
18		5.50	10.00	180.00	-10.00	0.00	3.54	14.04
19		6.00	10.00	180.00	-10.00	0.00	-1.46	14.04
20		6.50	10.00	180.00	-10.00	0.00	-6.46	14.04
21		7.00	10.00	180.00	-10.00	0.00	-11.46	14.04
22		7.50	10.00	180.00	-10.00	0.00	-16.46	14.04
23		8.00	10.00	180.00	-10.00	0.00	-21.46	14.04
24		8.50	10.00	180.00	-10.00	0.00	-26.46	14.04
25		9.00	10.00	180.00	-10.00	0.00	-31.46	14.04
26		9.50	10.00	180.00	-10.00	0.00	-36.46	14.04
27		10.00	10.00	180.00	-10.00	0.00	-41.46	14.04

=B7+dt

Time t has a step of dt, defined on cell C2.



This is a scatter graph: one numerical variable versus another numerical variable (y versus x, in this case)



copy these cells down...

copy these cells down...

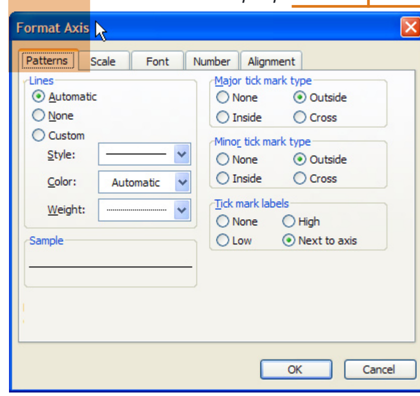
copy these cells down...

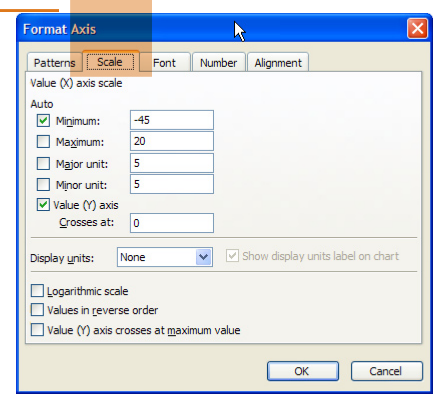
TIP: *Paint the cells that are independent from other cells and are used to give values to independent variables or parameters.*

angle	vx	vy
45.00	3.54	3.5
45.00		
0.00		

TIP: *to copy a cell down, place the mouse on the right down side of the cell and drag it*

TIP: *click on each axis to define its properties*





Analytical solution (explicit function) of the equations of motion

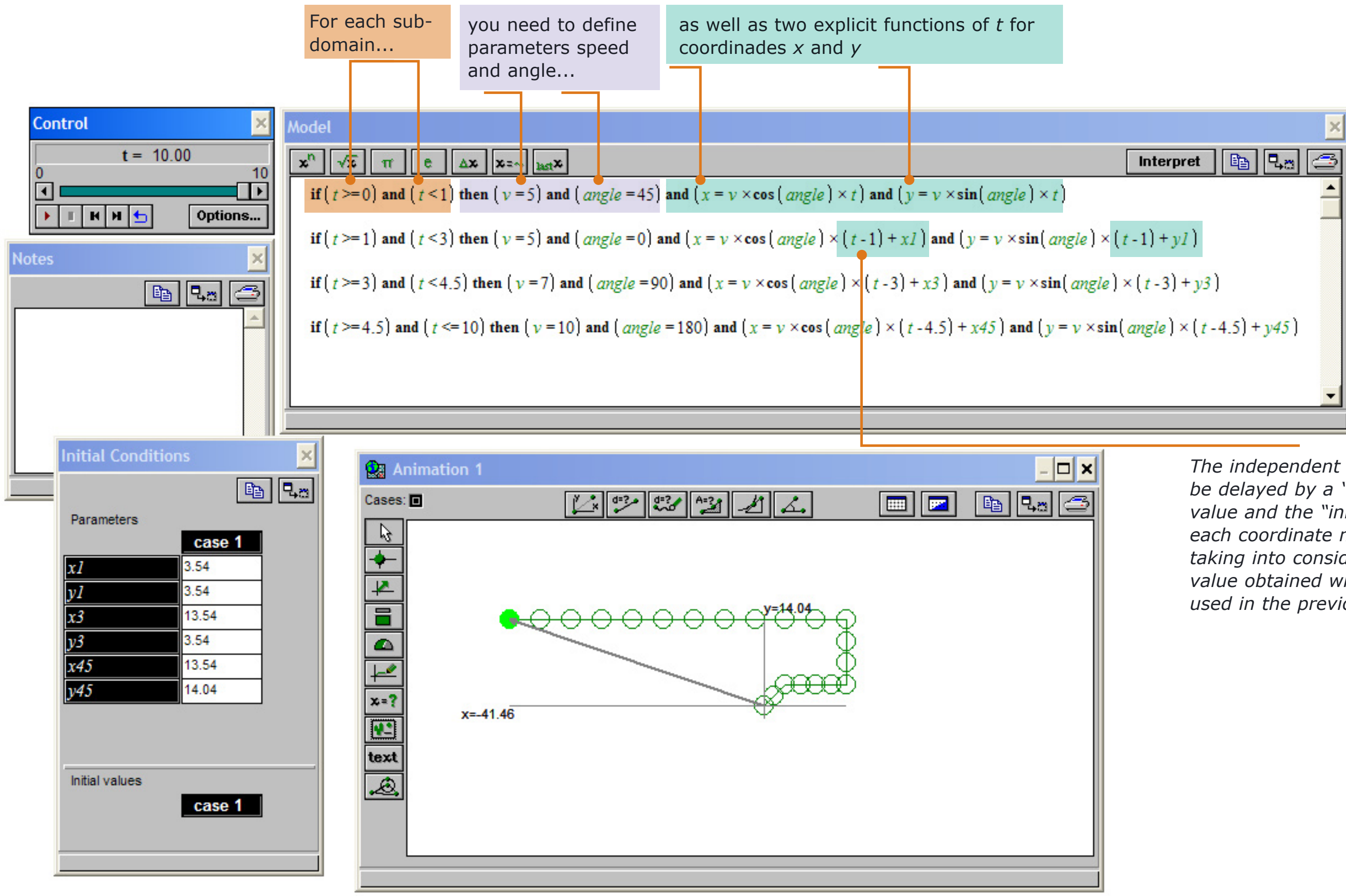
1. You don't need to write this model...

2. An analytical solution is an explicit function of the independent variable (*t*, in the case of the motion of the boat)...

3. For each sub-domain, it is necessary to define for the coordinates *x* and *y* a specific function of *t*, with the correct values of the parameters *v* and *angle* for that sub-domain...

4. The functions used in each sub-domain also need to take into consideration the initial values of *x* and *y* in each sub-domain of *t* as well as a convenient delay in *t* (the initial value of *t* in each sub-domain) in order to correctly describe the motion on the sub-domain.

5. What type of models (explicit functions/analytical solutions or iterative models/numerical solutions) do you think must be used in computer games to move characters and objects? Give your reasons.



The independent variable t must be delayed by a "convenient" value and the "initial" value of each coordinate must be set taking into consideration the value obtained with the functions used in the previous domain...

Controlling speed and direction of velocity...

1. Create the Model on the right (don't forget to check the Options... properties of the Control bar and the initial values for all variables in the Initial Conditions window).

2. The model states that, for each component,

3. Use the Vector button do create a vector with components v_x and v_y. This vector represents the magnitude and direction of the velocity of the particle (the "boat").

4. Run the Model. Use the mouse to change the magnitude and the direction of the boat...

5. Analyse the graphs of the x and y coordinates as functions of time. Do they make sense? Can you identify when the "boat" is travelling "fast"? And "slow"?